Enrollment No: ______________________ Exam Seat No:______________________

C.U.SHAH UNIVERSITY

Summer Examination-2022

Subject Name: Group Theory

Subject Code: 4SC05GRT1 **Branch: B.Sc. (Mathematics)**

Semester: 5 **Date: 25/04/2022** **Time: 11:00 To 02:00** **Marks: 70**

Instructions:

1. Use of Programmable calculator & any other electronic instrument is prohibited.
2. Instructions written on main answer book are strictly to be obeyed.
3. Draw neat diagrams and figures (if necessary) at right places.
4. Assume suitable data if needed.

Q-1 **Attempt the following questions:** **(14)**

a) For a group $(Z_5, +_5)$ then $O(4) =$ _______. **(01)**
(a) 1 (b) 2 (c) 4 (d) 5

b) Which of the following is not group ? **(01)**
(a) $(N,+)$ (b) $(Q,+)$ (c) $(R,+)$ (d) $(Z,+)$

c) Let G be a group of order n , for any $a \in G$ $a^n =$ ______ **(01)**
(a) a (b) a^2 (c) e (d) a^{-1}

d) The number of generators in group $(Z_6, +_6)$ is ________. **(01)**
(a) 1 (b) 2 (c) 3 (d) 4

e) $\sigma = (1\ 2\ 4\ 5\ 3) \in S_5$ is an ________ permutation. **(01)**
(a) odd (b) even (c) Identity (d) transposition

f) The permutation $\begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$ is equal to **(01)**
(a) $(1\ 3)(1\ 5)(2\ 4)$ (b) $(1)\ (2)\ (3)$ (c) $(1\ 3\ 5)(5\ 6)$ (d) $(1\ 4\ 2)\ (5\ 3)$

g) Let G be a finite group of order m and H be a subgroup of G of order n then_____. **(01)**
(a) m/n (b) n/m (c) $n = m$ always (d) all

h) Which of the following is true? **(01)**
(a) Every finite group is cyclic (b) Every cyclic group is abelian
(c) Every abelian group is cyclic (d) none of these

i) Every group of prime order is_______. **(01)**
(i) cyclic (ii) abelian (iii) sub –group (iv) Normal group

j) $(Z_4, +_4)$ is group then $2 +_4 3 =$_______. **(01)**
(a) 1 (b) 2 (c) 3 (d) 4

k) If $\mu = (1\ 2\ 3)(4\ 5)$ then $O(\mu) =$ _______. **(01)**
(a) 1 (b) 2 (c) 3 (d) 6

l) A cycle of length two is called **(01)**
(a) remainder (b) transposition (c) disjoint cycle (d) None

m) If H_1 and H_2 are two subgroups of G, then which following is also a subgroup of **(01)**

G.
(a) $H_1 \cap H_2$ (b) $H_1 \cup H_2$ (c) H_1H_2 (d) None

n) If $G = \{1, -1, i, -i\}$ is a multiplicative group then order of $-i$ is ______. **(01)**
(a) 1 (b) 2 (c) 3 (d) 4

Attempt any four questions from Q-2 to Q-8

Q-2 **Attempt all questions** **(14)**

a) Show that the set of all Integers form a group under the binary operation defined by as $a * b = a + b + 1 \quad \forall\, a, b \in Z$ **(05)**

b) Prove that for any group $(G,*)$ (i) the Identity element in $(G,*)$ is unique (ii) the inverse element in $(G,*)$ is uniqe **(05)**

c) Show that for group $(G,*)$ (i) $(a * b * c)^{-1} = c^{-1} * b^{-1} * a^{-1} \;\forall\, a, b, c \in G$ **(04)**

Q-3 **Attempt all questions** **(14)**

a) If H_1 and H_2 are two subgroups of group G then prove that $H_1 \cap H_2$ also subgroup of G **(05)**

b) Let G be a Group and let $a \in G$ then Prove that $N(a) = \{x \in G / xa = ax\}$ is a subgroup of G **(05)**

c) Let $(G,*)$ is group and $a, b \in G$ then the linear equation $a * x = b$ has unique solution in G **(04)**

Q-4 **Attempt all questions** **(14)**

a) For $\sigma, \mu \in S_5$ where $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix}$ and $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ then show that $(\sigma\mu)^{-1} = \mu^{-1}\sigma^{-1}$ **(05)**

b) Let G be a group and for $a \neq e$, $a^2 = e \quad \forall\, a \in G$ then show that G is abelian group **(05)**

c) Show that $\sigma_1 = \begin{pmatrix} 1234 \\ 2134 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 1234 \\ 1243 \end{pmatrix}$ are commute with each other **(04)**

Q-5 **Attempt all questions** **(14)**

a) Let G be a group and H be a subgroup of G then prove that $Ha = Hb \Leftrightarrow ab^{-1} \in H$ **(05)**

b) Suppose $o(a) = n$ for an element a in a group G. Then prove that **(05)**
(i) $o(a^p) \leq o(a), p \in Z$
(ii) $o(a^{-1}) = o(a)$
(iii) For a positive integer q with $(q, n) = 1$ then prove that $o(a^q) = o(a)$.

c) Let H be a subgroup of G and $a, b \in G$ then show that $a \in H \Leftrightarrow H = Ha$ **(04)**

Q-6 **Attempt all questions** **(14)**

a) Show that the set $\{1, -1, i, -i\}$ is cyclic group with respect to multiplication **(05)**

b) Prove that every cyclic group is an abelian but converse is not true **(05)**

c) Find the order of each elements in cyclic group $(Z_8, +)$ and also find all generators of z_8. **(04)**

Q-7 **Attempt all questions** **(14)**

a) Let $G = (\mathbb{R}, +)$ and $G' = (\mathbb{R}_+, \cdot)$, Let $f : G \rightarrow G'$ be defined as $f(x) = e^x, \;\forall\, x \in G$ then prove that f is an isomorphism between G and G'. **(05)**

b) Suppose $(G, \circ) \cong (G', *)$. Then prove that if G is commutative then G' is commutative **(05)**

c) Let $G = \{1, -1, i, -i\}$ and $H = \{1, -1\}$ then show that H is normal subgroup of G **(04)**

Q-8 **Attempt all questions** **(14)**

a) State and prove Caley's theorem **(07)**

b) State and prove Langrange's theorem **(07)**